



THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

Electricity Tracing in Systems With and Without Circulating Flows: Physical Insights and Mathematical Proofs

Citation for published version:

Achayuthakan, C, Dent, C, Bialek, J & Ongsakul, W 2010, 'Electricity Tracing in Systems With and Without Circulating Flows: Physical Insights and Mathematical Proofs', *IEEE Transactions on Power Systems*, vol. 25, no. 2, pp. 1078-1087. <https://doi.org/10.1109/TPWRS.2009.2037506>

Digital Object Identifier (DOI):

[10.1109/TPWRS.2009.2037506](https://doi.org/10.1109/TPWRS.2009.2037506)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Peer reviewed version

Published In:

IEEE Transactions on Power Systems

General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.



Electricity Tracing in Systems With and Without Circulating Flows: Physical Insights and Mathematical Proofs

Chira Achayuthakan, Chris J. Dent, *Member, IEEE*, Janusz W. Bialek, *Senior Member, IEEE*, and Weerakorn Ongsakul, *Member, IEEE*

Abstract—This paper provides new insights into the electricity tracing methodology, by representing the inverted tracing upstream and downstream distribution matrices in the form of matrix power series and by applying linear algebra analysis. The n th matrix power represents the contribution of each node to power flows in the other nodes through paths of length exactly n in the digraph of flows. Such a representation proves the link between graph-based and linear equation-based approaches for electricity tracing. It also makes it possible to explain an earlier observation that circulating flows, which result in a cyclic directed graph of flows, can be detected by appearance of elements greater than one on the leading diagonal of the inverted tracing distribution matrices. Most importantly, for the first time a rigorous mathematical proof of the invertibility of the tracing distribution matrices is given, along with a proof of convergence for the matrix power series used in the paper; these proofs also allow an analysis of the conditioning of the tracing distribution matrices. Theoretical results are illustrated throughout using simple network examples.

Index Terms—Power system economics, power transmission economics.

I. INTRODUCTION

EVER since electricity supply systems throughout the world started to be liberalized in the early 1990s, transmission pricing has proved to be a contentious issue. The main reason for this is that a proper transmission pricing regime should satisfy a number of requirements which are in tension [1]. Two main apparently conflicting requirements are that competition should be promoted by presenting the network user with a predictable, stable and practically applicable framework of charges, and that transmission prices should provide signals toward the efficient use, operation and expansion of the network. Many transmission pricing methodologies have been proposed in the literature, with each of them putting a different emphasis on one or other of the requirements.

Manuscript received April 15, 2009; revised July 08, 2009, September 16, 2009. First published February 22, 2010; current version published April 21, 2010. This work was supported in part by the Queen Sirikit HRD Project, Energy Policy and Planning Office (EPPO), Ministry of Energy, Thailand. The work of C. J. Dent was supported through the EPSRC Supergen V, UK Energy Infrastructure (AMPerES) grant in collaboration with UK electricity network operators working under Ofgem's Innovation Funding Incentive scheme—full details on <http://www.supergen-amperes.org/>. Paper no. TPWRS-00273-2009.

C. Achayuthakan and W. Ongsakul are with Energy Field of Study, Asian Institute of Technology (AIT), Pathumthani 12120, Thailand (e-mail: st104252@ait.ac.th; Ongsakul@ait.ac.th).

J. W. Bialek and C. J. Dent are with the School of Engineering and Computing Sciences, Durham University, Durham DH1 2PE, U.K. (e-mail: Janusz.Bialek@durham.ac.uk; chris.dent@durham.ac.uk)

Digital Object Identifier 10.1109/TPWRS.2009.2037506

One of the more popular approaches to transmission pricing is based on application of the tracing methodology [2]. Tracing is a simple and transparent methodology that attempts to trace the flow of electricity in the network from individual generators to individual loads by following the directed graph (digraph) of flows, assuming that at any node the inflows are proportionally distributed among the outflows. The proportionality assumption is intuitively reasonable yet it cannot be proved. However, it can be rationalized using cooperative game theory and information theory, by showing that the proportionality assumption results in the optimal cost allocation regardless of the form of the cost function [3]. Electricity tracing is used in the Northeast China power system [4] and it is being considered as a candidate for Inter-TSO Compensation in Europe [5]. Electricity tracing tends to result in lower price differentials than marginal pricing, and hence it might be more easily accepted in countries with power industries in a transitional phase [6]. It should be noted that first attempts to develop a tracing methodology were undertaken by the New Zealand utility TransPower in the late 1980s; however, no publications are publicly available.

Two main approaches to electricity tracing have been proposed in the past: an approach based on the solution of simultaneous linear equations [2], [7], and a graph-based approach which results in an iterative algorithm [8], [9]. The linear equations-based algorithm is able to calculate contributions of all individual nodes even in the presence of circulating flows which create cycles in the digraph of flows. On the other hand the graph-based algorithm is not able to differentiate between the contributions of any node in the cycle [10]. Other variations of the tracing methodology have also been proposed, e.g., [11] or [6], where generalized tracing was formulated as a linear constrained multi-commodity network flow optimization problem.

In this paper the inverted upstream and downstream tracing distribution matrices, originally derived in [2], are represented in the form of matrix power series. This allows us to make three important contributions. Firstly, it provides an additional insight into the tracing methodology, by explaining the link between the linear equation-based and graph-based approaches to tracing. The n th matrix power represents the contribution of each node to power flows in the other nodes through paths of length exactly n , i.e., through $n - 1$ intermediate nodes in the digraph of flows, hence making explicit the link between the two main approaches to tracing. Secondly, the matrix power series formulation allows us to explain an earlier observation [12] that circulating flows, which cause cycles in the digraph of flows, can be detected by the appearance of elements greater than one on the leading diagonal of the inverted tracing distribution matrices.

The third, and perhaps most important, contribution of the paper is in analyzing the existence, uniqueness of solution and conditioning of the matrices which must be inverted in the tracing problem. We provide, for the first time, a rigorous mathematical proof of invertibility of the tracing distribution matrices in systems with and without circulating flows. We demonstrate that a unique tracing solution exists for any physically valid power flow and that the tracing problem has no solution only in some special cases when subsystems in an interconnected network are aggregated to form supernodes.

The results are important both from theoretical and practical points of view, as they improve understanding of, and confidence in, the tracing methodology. The considerations are illustrated using simple network examples.

II. LOSSLESS NETWORK WITHOUT CIRCULATING FLOWS

Tracing of power flows in a network can be executed upstream, i.e., from the loads to the generators, or downstream, i.e., from the generators to the loads [2]. To concentrate attention, in this paper mainly the lossless downstream version of the methodology will be considered. Clearly, the upstream version can be considered in a similar way, and the influence of transmission losses could also be included easily by considering gross or net versions of the methodology [2]. Gross flows will include all transmission losses accumulated as tracing proceeds downstream from the generators to the loads, while net flows will exclude all the transmission losses as tracing proceeds upstream from the loads to the generators. The methodology can also be extended easily to trace the flow of reactive power [2] by (e.g.) establishing additional, fictitious, “line nodes” responsible for the reactive power loss in each line. Note that tracing reactive power independently from active power has to be treated with care because a substantial portion of the reactive power flows is due to reactive losses caused by active power flows.

Alternative approaches to dealing with losses and reactive power have been suggested, e.g., in [7]. Modification of the methodology presented in this paper to employ any of those methods is relatively simple, and will not be discussed here due to lack of space.

Tracing can be applied to power flows in a full line-by-line transmission network model, or alternatively it can be used to assess how cross-border trades are distributed in a reduced model of an interconnected network; in the latter case, each subsystem is represented by a supernode, whose generation and demand are equal to the total generation and demand in the subsystem [13]. The examples used in this paper illustrate the latter application as it is far easier to construct simple examples and incorporate circulating flows in a reduced model of an interconnected network rather than in a full network model. However the methodology presented in this paper is equally well applicable to tracing electricity in a full model of a single network with each line explicitly represented.

A. Topological Properties of Tracing Matrices

The tracing methodology was originally derived in [2] using the proportional sharing principle in order to derive the downstream and upstream distribution matrices \mathbf{A}_d and \mathbf{A}_u . Here the

same matrices will be derived using formal matrix manipulations. The main aim will be to show a link between the adjacency matrix and the downstream and upstream distribution matrices introduced in [2].

Consider a network consisting of $nbus$ nodes and m branches and define \mathbf{P} as $(nbus \times 1)$ vector of nodal flows (i.e., the sum of nodal inflows or outflows including local generation and demand, respectively), \mathbf{P}_G as $(nbus \times 1)$ vector of nodal generations, \mathbf{P}_D as $(nbus \times 1)$ vector of nodal demands, and \mathbf{F} as $(m \times 1)$ vector of branch flows. Fig. 1 shows an example of a network with power flows, generations and demands where

$$\begin{aligned}\mathbf{P} &= [20 \ 55 \ 25 \ 30 \ 25 \ 20]^T \\ \mathbf{P}_G &= [20 \ 35 \ 0 \ 10 \ 0 \ 0]^T \\ \mathbf{P}_D &= [0 \ 0 \ 10 \ 15 \ 20 \ 20]^T \\ \mathbf{F} &= [20 \ 15 \ 25 \ 15 \ 5 \ 15 \ 10 \ 5]^T.\end{aligned}$$

The $(m \times nbus)$ incidence matrix \mathbf{B} is

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad (1)$$

where 1 corresponds to the start of a line (as denoted by the direction of flow) and -1 denotes the end of a line. This matrix can be split into matrix \mathbf{B}_u consisting of -1 's and \mathbf{B}_d consisting of 1's, i.e.

$$\begin{aligned}\mathbf{B}_u &= \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \\ \mathbf{B}_d &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.\end{aligned} \quad (2)$$

The adjacency matrix \mathbf{C} of a digraph is defined as $(nbus \times nbus)$ matrix with $[\mathbf{C}]_{ij} = 1$ if there is a flow from node i to node j . The adjacency matrix can be calculated as $\mathbf{C} = -\mathbf{B}_d^T \mathbf{B}_u$, and for the network in Fig. 1 it is

$$\mathbf{C} = -\mathbf{B}_d^T \mathbf{B}_u = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (3)$$

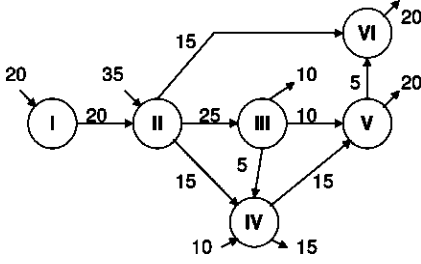


Fig. 1. Simple example system.

Let us now define $(nbus \times nbus)$ matrix \mathbf{F}_d such that its (i, j) element is equal to the flow in line $i - j$ towards node j (i.e., downstream). \mathbf{F}_d can be calculated as

$$\mathbf{F}_d = -\mathbf{B}_d^T \text{diag}(\mathbf{F}) \mathbf{B}_u \quad (4)$$

where the operator $\text{diag}(\cdot)$ denotes a diagonal matrix constructed from a vector. Clearly \mathbf{F}_d has the same structure as the adjacency matrix \mathbf{C} . For the network shown in Fig. 1

$$\mathbf{F}_d = \begin{bmatrix} 0 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 25 & 15 & 0 & 15 \\ 0 & 0 & 0 & 5 & 10 & 0 \\ 0 & 0 & 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (5)$$

The vector of nodal flows \mathbf{P} can be defined as the sum of nodal outflows or inflows

$$\mathbf{P} = \mathbf{P}_D + \mathbf{F}_d \mathbf{1} \quad \text{or} \quad \mathbf{P} = \mathbf{P}_G + \mathbf{F}_d^T \mathbf{1} \quad (6)$$

where $\mathbf{1}$ is $(nbus \times 1)$ vector of ones. Substituting (4) into (6)

$$\begin{aligned} \mathbf{P}_D &= \mathbf{P} - \mathbf{F}_d \mathbf{1} = \mathbf{P} + \mathbf{B}_d^T \text{diag}(\mathbf{F}) \mathbf{B}_u \mathbf{1} \\ &= [\mathbf{I} + \mathbf{B}_d^T \text{diag}(\mathbf{F}) \mathbf{B}_u (\text{diag} \mathbf{P})^{-1}] \mathbf{P} = \mathbf{A}_d \mathbf{P} \end{aligned} \quad (7)$$

$$\begin{aligned} \mathbf{P}_G &= \mathbf{P} - \mathbf{F}_d^T \mathbf{1} = \mathbf{P} + \mathbf{B}_u^T \text{diag}(\mathbf{F}) \mathbf{B}_d \mathbf{1} \\ &= [\mathbf{I} + \mathbf{B}_u^T \text{diag}(\mathbf{F}) \mathbf{B}_d (\text{diag} \mathbf{P})^{-1}] \mathbf{P} = \mathbf{A}_u \mathbf{P} \end{aligned} \quad (8)$$

assuming that nodes with no nodal flows, i.e., those for which $P_i = 0$, have been removed from the digraph so that $(\text{diag} \mathbf{P})^{-1}$ exists.

Matrices \mathbf{A}_d and \mathbf{A}_u are referred to as the *downstream* and *upstream distribution matrices* and are equal to

$$\begin{aligned} \mathbf{A}_d &= \mathbf{I} + \mathbf{B}_d^T \text{diag}(\mathbf{F}) \mathbf{B}_u (\text{diag} \mathbf{P})^{-1} \\ \mathbf{A}_u &= \mathbf{I} + \mathbf{B}_u^T \text{diag}(\mathbf{F}) \mathbf{B}_d (\text{diag} \mathbf{P})^{-1}. \end{aligned} \quad (9)$$

Comparing (3) with (9) shows that \mathbf{A}_d has the same structure as the adjacency matrix \mathbf{C} , with addition of a diagonal, while the structure of \mathbf{A}_u is the transpose of the structure of \mathbf{A}_d .

Obviously \mathbf{A}_d and \mathbf{A}_u can also be formed directly from the line flows, without using the formal matrix manipulations shown in (9). Their elements are simply equal to

$$\begin{aligned} [\mathbf{A}_d]_{ij} &= \begin{cases} 1 & \text{for } i = j \\ -|P_{ji}|/P_j & \text{for } j \in \alpha_i^d \end{cases} \\ [\mathbf{A}_u]_{ij} &= \begin{cases} 1 & \text{for } i = j \\ -|P_{ji}|/P_j & \text{for } j \in \alpha_i^u \end{cases} \end{aligned} \quad (10)$$

where P_{ij} is the flow in the line linking nodes i and j , P_j is the nodal power at node j , α_i^d is the set of downstream nodes supplied directly from node i and α_i^u is the set of upstream nodes supplying directly node i . For the network shown in Fig. 1, we see

$$\mathbf{A}_d = \begin{bmatrix} 1 & -20/55 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -15/30 & 0 & -15/20 \\ 0 & 0 & 1 & -5/30 & -10/25 & 0 \\ 0 & 0 & 0 & 1 & -15/25 & 0 \\ 0 & 0 & 0 & 0 & 1 & -5/20 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (11)$$

$$\mathbf{A}_u = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -25/55 & 1 & 0 & 0 & 0 \\ 0 & -15/55 & -5/25 & 1 & 0 & 0 \\ 0 & 0 & -10/25 & -15/30 & 1 & 0 \\ 0 & -15/55 & 0 & 0 & -5/25 & 1 \end{bmatrix}. \quad (12)$$

Equations (7) and (8) make it possible to trace how power flows in the network from individual generators or to individual demands using

$$\mathbf{P} = \mathbf{A}_d^{-1} \mathbf{P}_D \quad \text{or} \quad \mathbf{P} = \mathbf{A}_u^{-1} \mathbf{P}_G \quad (13)$$

assuming that \mathbf{A}_d^{-1} and \mathbf{A}_u^{-1} exist. The question of invertibility of \mathbf{A}_d and \mathbf{A}_u will be considered later.

Note that (13) has been derived by mathematical manipulation of the Kirchhoff's Current Law expressed by (6). Hence (13) does not represent a causal relationship between nodal powers and generations and demands, but it can still be used for transmission cost allocation purposes.

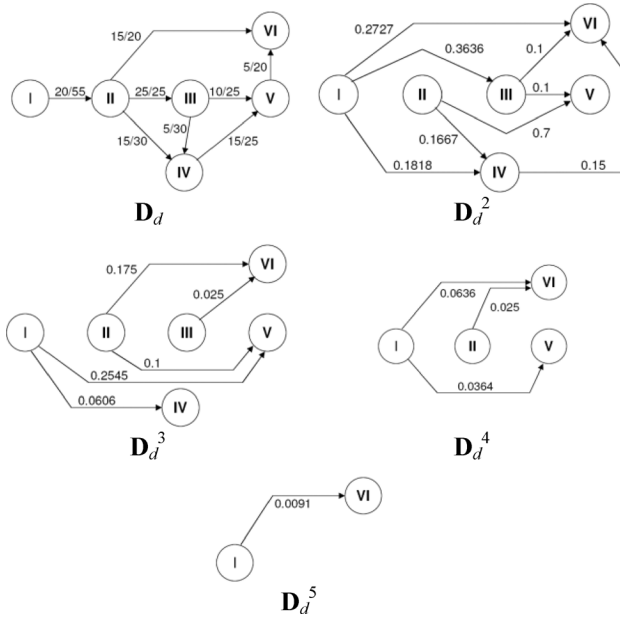
The proportion of the nodal flow P_i coming from the local generation P_{Gi} is equal to P_{Gi}/P_i . Hence the nodal generation P_{Gi} can be expressed, using the first equation of (13), as a linear combination of components supplied to individual demands P_{Dk}

$$P_{Gi} = \frac{P_{Gi}}{P_i} \sum_{k=1}^{nbus} [\mathbf{A}_d^{-1}]_{ik} P_{Dk}. \quad (14)$$

Similarly, as the proportion of the nodal flow P_i coming from power P_{ji} flowing in the line $j-i$ supplying it is P_{ji}/P_i , branch flows can be expressed as a linear combination of components supplied to individual demands

$$P_{ji} = \frac{P_{ji}}{P_i} \sum_{k=1}^{nbus} [\mathbf{A}_d^{-1}]_{ik} P_{Dk} \quad \text{for } j \in \alpha_i^u. \quad (15)$$

The influence of transmission losses can be accounted for by considering gross and net network flows [2]. To obtain the net flows, the vector \mathbf{F} used to calculate \mathbf{A}_d in (9) must be replaced by the vector of flows at the receiving end of each line. To obtain the gross flows, the flow vector \mathbf{F} used to calculate \mathbf{A}_u in (9) must be replaced by the vector of flows at the sending end of each line.

Fig. 2. Digraphs of \mathbf{D}_d^n for $n = 1$ to 5.

B. Direct and Indirect Contributions of Nodal Power to Other Nodes

Electricity tracing as defined by (7) and (8) is based on solving linear equations, while the graph-based tracing methodology [8], [9] is based on finding all possible paths between any two nodes in the digraph. Now the link between the graph-based and the linear equations-based approaches to electricity tracing will be explored by using a well-known property of the adjacency matrix \mathbf{C} , namely that the value of element (i, j) of the power n of the adjacency matrix, $[\mathbf{C}^n]_{ij}$, gives the number of paths of length n between nodes i and j [14]. Due to space limitations we will concentrate on the downstream distribution matrix \mathbf{A}_d , but all the properties discussed in this paper can equally well be applied to matrix \mathbf{A}_u using symmetry arguments.

Now define matrix \mathbf{D}_d as

$$\mathbf{D}_d = \mathbf{I} - \mathbf{A}_d. \quad (16)$$

Obviously $\mathbf{D}_d = -\mathbf{B}_d^T \text{diag}(\mathbf{F}) \mathbf{B}_u (\text{diag} \mathbf{P})^{-1}$, so \mathbf{D}_d has the same structure as the adjacency matrix $\mathbf{C} = -\mathbf{B}_d^T \mathbf{B}_u$, and its (i, j) th element expresses the share of P_j which is supplied directly from node i (i.e., through line $i - j$)

$$\mathbf{D}_d = \begin{bmatrix} 0 & 20/55 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 15/30 & 0 & 15/20 \\ 0 & 0 & 0 & 5/30 & 10/25 & 0 \\ 0 & 0 & 0 & 0 & 15/25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5/20 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (17)$$

The digraph corresponding to matrix \mathbf{D}_d is shown in the top-left diagram in Fig. 2. Comparison with Fig. 1 shows that each line flow has been divided by the value of nodal power at the receiving end of the line in order to express a share in the nodal flow. For simplicity, generations and demands have

been omitted in Fig. 2. The sum of the nodal shares for each node, i.e., the sum of elements in each column j , must add up to 1 when there is no generation connected to a node; see, e.g., nodes III, V and VI in Fig. 2. On the other hand, when there is a local generator connected to a node, e.g., in nodes II and IV in Fig. 2, this generator also contributes to the nodal power. It follows that the shares provided by other nodes, i.e., the elements in a column corresponding to the node in question, add up to less than one. This property of \mathbf{D}_d is important for the proof of invertibility of the tracing distribution matrices—see Section IV.

As mentioned earlier, the value of element (i, j) of the n th power of the adjacency matrix, $[\mathbf{C}^n]_{ij}$, gives the number of paths of length n between nodes i and j . Note that \mathbf{D}_d has the same structure as the adjacency matrix \mathbf{C} , but its elements are equal to the share of P_j that is supplied directly from node i . Hence it follows that the value of element (i, j) of the n th power of \mathbf{D}_d , $[\mathbf{D}_d^n]_{ij}$, will give the share of P_j which is supplied from node i through paths of length exactly n between nodes i and j . To understand this let us consider matrix \mathbf{D}_d^2 in (18) at the bottom of the next page.

The element $[\mathbf{D}_d^2]_{ij} = \sum_k [\mathbf{D}_d]_{ik} [\mathbf{D}_d]_{kj}$ expresses the proportion of the nodal power at node j that is supplied from node i indirectly through any single intermediate node k . For example element $[\mathbf{D}_d^2]_{2,5}$ shows that node V is supplied from node II indirectly via nodes III and IV and the respective shares of the supply to V are $1 \times 10/25$ and $(15/30)(15/25)$. The latter share is made up of 15/30, due to connection II–IV, multiplied by (15/25) due to connection IV–V. A digraph corresponding to matrix \mathbf{D}_d^2 is also shown in Fig. 2. Clearly all the direct connections (i.e., the paths of length one) between any two nodes have been eliminated, and the resulting elements correspond to the paths of length two between any two nodes.

Similarly let us now consider matrices \mathbf{D}_d^3 , \mathbf{D}_d^4 , and \mathbf{D}_d^5 :

$$\mathbf{D}_d^3 = \begin{bmatrix} 0 & 0 & 0 & 0.0606 & 0.2545 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0.175 \\ 0 & 0 & 0 & 0 & 0 & 0.025 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D}_d^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.0364 & 0.0636 \\ 0 & 0 & 0 & 0 & 0 & 0.025 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D}_d^5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.0091 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (19)$$

The corresponding digraphs are shown in Fig. 2. Element $[\mathbf{D}_d^3]_{ij}$ and the corresponding digraph show the proportion of nodal power at node j that is supplied from node i indirectly through paths of length 3, i.e., through

two intermediate nodes. For example $[D_d^3]_{1,5}$ shows that node V is supplied from node I by two parallel routes: via nodes II and III and via nodes II and IV. The share is $(20/55)(15/30)(15/25) + (20/55)(25/25)(10/25) = 0.2545$. The corresponding digraph shows that the paths of length two (D_d^2 in Fig. 2) have been eliminated. A similar interpretation can be associated with matrices D_d^4 and D_d^5 and their corresponding digraphs.

Generally, an element $[D_d^n]_{ij}$ and the corresponding digraph show the proportion of the nodal power at node j that is supplied from node i indirectly through paths of length n , i.e., when power flows through $(n - 1)$ intermediate nodes.

The maximum nonzero matrix power has n equal to the height of the original digraph, i.e., the longest path between any two nodes. In the example system, Fig. 1, the height of the graph is 5 as the longest path is (I–II–III–IV–V–VI). Hence the corresponding D_d^5 has only one nonzero element, and D_d^6 is obviously zero.

Now let us consider matrix T_d :

$$T_d = I + D_d + D_d^2 + D_d^3 + D_d^4 + D_d^5 + \dots \quad (20)$$

In our example, we see (21) at the bottom of the page.

Element $[T_d]_{ij}$ shows the share of the nodal power at node j which is supplied from node i through all intermediate connections. For example $[T_d]_{1,6}$ consist of 0 from D_d , $(20/55)(15/20) = 0.2727$ from D_d^2 , 0 from D_d^3 , 0.0636 from D_d^4 , and 0.0091 from D_d^5 . This means that node I supplies node VI in three ways: 0.2727 of the nodal power at node VI comes through paths including 1 intermediate node, 0.0636 is supplied through paths including 2 intermediate nodes and 0.0091 from paths including 4 intermediate nodes. In total, node I supplies $(0.2727 + 0.0636 + 0.0091) = 0.3455$ of the nodal power at node VI. The diagonal elements of T_d are all one, as all power flowing through a node must obviously come from that node.

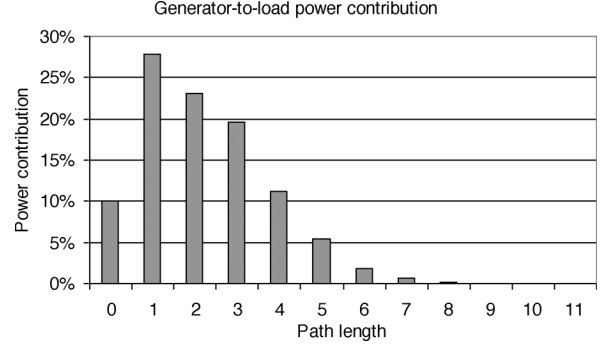


Fig. 3. Power supplied over different path lengths for 118-node IEEE system.

It is interesting to study how far power travels in a typical network, or in other words what the maximum path length (height of the digraph) is. Fig. 3 shows the shares of power supplied from individual generators to individual loads over paths of different lengths obtained for the standard 118-node IEEE test system [15]. The system has 186 lines, 19 generator buses, and 99 load points (ten out of them are on buses having a generator). Power supplied by a generator to a load over path length n can be quantified using (14) with D_d^n replacing A_d^{-1} . The shares in Fig. 3 are expressed as percentages of the system load of 4242 MW. Zero path length corresponds to a generator connected to a load at the same bus. The maximum path length in the network is 11 but 92% of power is supplied over path lengths not exceeding 4 and 99% of power is supplied over path lengths not exceeding 6. This shows that (as identified by tracing) generally power does not travel very far in a network.

C. Connection With the Original Tracing Distribution Matrices

We will now show the connection between the downstream distribution matrix A_d and matrix T_d introduced in this paper. A connection between the upstream distribution matrix A_u and

$$D_d^2 = \begin{bmatrix} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} & \text{VI} \\ \begin{bmatrix} 0 & 0 & (20/55) \times 1 & (20/55)(15/30) & 0 & (20/55)(15/20) \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 1 \times (5/30) & 1 \times (10/25) + (15/30)(15/25) & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & (5/30)(15/25) & (10/25)(5/20) \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & (15/25)(5/20) \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix} \begin{matrix} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \\ \text{V} \\ \text{VI} \end{matrix} \quad (18)$$

$$T_d = \begin{bmatrix} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} & \text{VI} \\ \begin{bmatrix} 1 & 0.3636 & 0.3636 & 0.2424 & 0.2909 & 0.3455 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 1 & 0.6667 & 0.8 & 0.95 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 & 0.1667 & 0.5 & 0.125 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0.6 & 0.15 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0.25 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{bmatrix} \begin{matrix} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \\ \text{V} \\ \text{VI} \end{matrix} \quad (21)$$

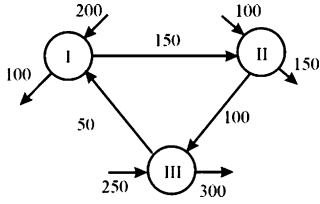


Fig. 4. Simple three-node network with circulating flow.

matrix \mathbf{T}_u can be derived in a similar way (\mathbf{T}_u is formed in a similar way as \mathbf{T}_d but looking at upstream shares.)

Pre-multiplying both sides of (20) by \mathbf{D}_d gives

$$\mathbf{D}_d \mathbf{T}_d = \mathbf{D}_d + \mathbf{D}_d^2 + \mathbf{D}_d^3 + \mathbf{D}_d^4 + \cdots = \sum_{n=1}^{\infty} \mathbf{D}_d^n. \quad (22)$$

Note that $\mathbf{D}_d^n = 0$ for n greater than the height of the graph. Subtracting (22) from (20) gives

$$(\mathbf{I} - \mathbf{D}_d) \mathbf{T}_d = \mathbf{I} \quad \text{or} \quad \mathbf{T}_d = (\mathbf{I} - \mathbf{D}_d)^{-1}. \quad (23)$$

As $\mathbf{A}_d = \mathbf{I} - \mathbf{D}_d$ —see (16)—we finally obtain that

$$\mathbf{T}_d = \mathbf{I} + \sum_{n=1}^{\infty} \mathbf{D}_d^n = (\mathbf{I} - \mathbf{D}_d)^{-1} = (\mathbf{A}_d)^{-1} \quad (24)$$

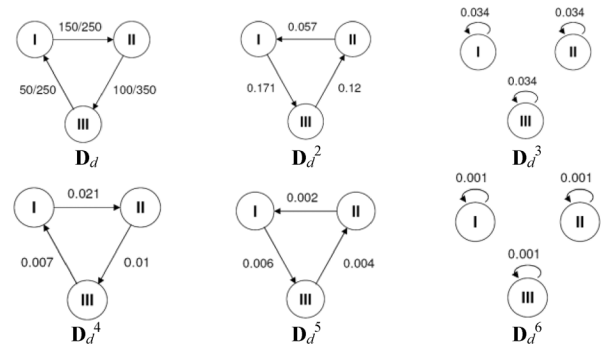
assuming that \mathbf{A}_d is nonsingular and hence invertible. Equation (24) can also be derived using matrix Taylor series expansion of $(\mathbf{I} - \mathbf{D}_d)^{-1}$ [16].

Equation (24) proves the link between the graph-based and linear equation-based approaches to electricity tracing, by showing that each element of matrix $(\mathbf{A}_d)^{-1} = \mathbf{T}_d$ is equal to the sum of contributions to the nodal flows traced back to different nodes in the digraph.

The inverted tracing distribution matrix $(\mathbf{A}_d)^{-1} = \mathbf{T}_d$ can be calculated in two ways: either by adding consecutive powers of matrix \mathbf{D}_d as in (20), or by inverting $\mathbf{A}_d = (\mathbf{I} - \mathbf{D}_d)$. It should be emphasized that we do not suggest using the matrix power series form for practical implementation of tracing, as inverting \mathbf{A}_d is obviously computationally more efficient. The point is that the matrix power series formulation provides a powerful insight into the physical meaning of the inverted downstream distribution matrix $(\mathbf{A}_d)^{-1}$. This finding will be further explored in the rest of the paper.

III. LOSSLESS NETWORK WITH CIRCULATING FLOWS

Usually power flows in a network can be represented by an acyclic digraph, such as that shown in Fig. 1. However there are some instances when the digraph of flows contains cycles because of the presence of circulating flows. Such flows may appear due to the use of multiple uncoordinated phase angle regulators [16], or when network nodes in some areas are aggregated for transmission pricing purposes [13]. Cycles are more common in reactive power flows when they may appear due to off-nominal voltage ratios of transformers [16]. The linear equations-based tracing algorithm is able to calculate contributions of all individual nodes even in the presence of circulating flows which create cycles in the digraph of flows. On the other

Fig. 5. Digraphs corresponding to consecutive powers of matrix \mathbf{D}_d .

hand the graph-based algorithm is not able to differentiate between the contributions of different nodes in a cycle [10]. In this section we will provide a further insight into this problem by capitalizing on the matrix power representation of the inverted tracing distribution matrices.

Fig. 4 shows a simple example of a system with a circulating flow. Matrix \mathbf{D}_d and its powers are shown below while the corresponding digraphs are shown in Fig. 5:

$$\begin{aligned} \mathbf{D}_d &= \begin{bmatrix} 0 & 150/250 & 0 \\ 0 & 0 & 100/350 \\ 50/250 & 0 & 0 \end{bmatrix} \\ \mathbf{D}_d^2 &= \begin{bmatrix} 0 & 0 & 0.171 \\ 0.057 & 0 & 0 \\ 0 & 0.120 & 0 \end{bmatrix} \\ \mathbf{D}_d^3 &= \begin{bmatrix} 0.034 & 0 & 0 \\ 0 & 0.034 & 0 \\ 0 & 0 & 0.034 \end{bmatrix} \\ \mathbf{D}_d^4 &= \begin{bmatrix} 0 & 0.021 & 0 \\ 0 & 0 & 0.01 \\ 0.007 & 0 & 0 \end{bmatrix} \\ \mathbf{D}_d^5 &= \begin{bmatrix} 0 & 0 & 0.006 \\ 0.002 & 0 & 0 \\ 0 & 0.004 & 0 \end{bmatrix} \\ \mathbf{D}_d^6 &= \begin{bmatrix} 0.001 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 0.001 \end{bmatrix}. \end{aligned} \quad (25)$$

Each matrix power \mathbf{D}_d^n shows the nodal contributions corresponding to the connections of path length n between any two network nodes. When circulating flows are absent, the height h of a digraph is finite, and hence all the connections are eliminated after h steps so that $\mathbf{D}_d^n = 0$ for $n > h$. When circulating flows are present, the height of the graph is infinite and the algorithm finds connections of any length, however large. In the case considered here, the length of the cycle is 3 as it consists of three nodes. For matrices \mathbf{D}_d and \mathbf{D}_d^2 the eliminations initially appear similar to the case without circulating flows; however, the number of nonzero elements in the matrix is not decreasing. For path length 3 (equal to the cycle length), \mathbf{D}_d^3 represents the contribution of each node to itself through one complete cycle. This is shown by the appearance of nonzero values on the diagonal of \mathbf{D}_d^3 , and by loops appearing in the digraph. Starting

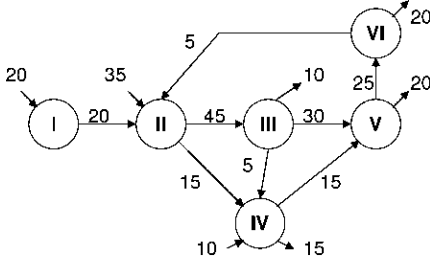


Fig. 6. Six-node lossless network with circulating flows.

from the fourth step the pattern repeats itself, but with matrices \mathbf{D}_d^n getting smaller as n increases, until for \mathbf{D}_d^8 all the elements are smaller than 0.001. The final value of matrix \mathbf{T}_d can be calculated either by adding more matrices in the series, or in one step by inverting $(\mathbf{I} - \mathbf{D}_d)$

$$\mathbf{T}_d = \begin{bmatrix} 1.0355 & 0.6213 & 0.1775 \\ 0.0592 & 1.0355 & 0.2959 \\ 0.2071 & 0.1243 & 1.0355 \end{bmatrix}. \quad (26)$$

A. Detection of Circulating Flows

Circulating flows can be easily identified by inspection in a small network but their identification in large networks may not be easy. It was suggested in [12] that nodes participating in circulating flows can be identified through the appearance of elements greater-than-one on the leading diagonal of the inverted upstream and downstream distribution matrices: $(\mathbf{A}_u)^{-1} = \mathbf{T}_u$ and $(\mathbf{A}_d)^{-1} = \mathbf{T}_d$. Here we will provide a physical justification for this observation based on the inspection of the matrices \mathbf{D}_d^n .

Each matrix power \mathbf{D}_d^n shows nodal contributions corresponding to indirect connection between any two nodes through paths of length n . In the absence of circulating flows, the diagonal elements of \mathbf{D}_d^n are all zero, as it is impossible to trace back a connection from a node to itself in an acyclic digraph. However, in the presence of circulating flows the digraph has cycles, and any node involved in a cycle can be traced back to itself through other nodes. In the example studied, Fig. 4, nonzero diagonal elements appeared in \mathbf{D}_d^3 because the length of the cycle is 3. Hence for path length of 3, the algorithm calculates the contribution of each node to itself. Obviously nonzero diagonal elements of \mathbf{D}_d^n cause the diagonal elements of $\mathbf{T}_d = (\mathbf{A}_d)^{-1} = \mathbf{I} + \sum_{n=1}^{\infty} \mathbf{D}_d^n$ to be greater than one, as observed originally in [12].

B. Example of Interacting Flows

Now let us assume that the interconnected network shown previously in Fig. 1 contains interacting circulating flows (which cause cycles to appear in the digraph) as shown in Fig. 6. There are three interacting cycles in the digraph: (II–IV–V–VI–II), (II–III–V–VI–II), and (II–III–IV–V–VI–II). The length of the first two cycles is 4 while the length of the last cycle is 5.

The consecutive powers \mathbf{D}_d^n can be calculated as before with the corresponding digraphs shown in Fig. 7 for $n = 1, 4, 5$, and 8. The interpretation of the results is similar to the case shown in Fig. 4. Because the length of the shortest cycle is 4, nonzero elements start to appear on the main diagonal of the

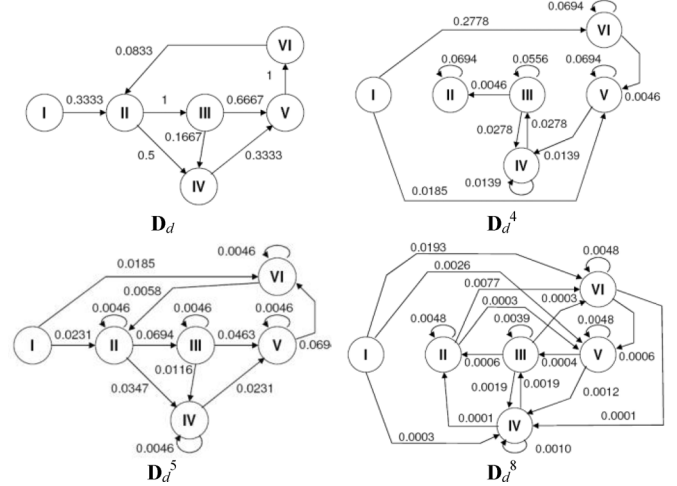


Fig. 7. Digraphs of \mathbf{D}_d^n ($n = 1, 4, 5$, and 8) for the six-node system with circulating flows.

matrices starting from \mathbf{D}_d^4 . This is illustrated in the digraphs by appearance of local loops in the nodes involved in the cycles. The consecutive powers \mathbf{D}_d^n tend towards zero for high n .

\mathbf{T}_d can be of course calculated directly as

$$\mathbf{T}_d = (\mathbf{I} - \mathbf{D}_d)^{-1}$$

	I	II	III	IV	V	VI	
=	1	0.36	0.36	0.24	0.32	0.32	I
	0	1.08	1.08	0.72	0.96	0.96	II
	0	0.065	1.065	0.21	0.78	0.78	III
	0	0.03	0.03	1.02	0.36	0.36	IV
	0	0.09	0.09	0.06	1.08	1.08	V
	0	0.09	0.09	0.06	0.08	1.08	VI

Node I does not participate in any of the cycles so the corresponding diagonal element of \mathbf{T}_d is one. As nodes II, V, VI participate in all three cycles (II–IV–V–VI–II), (II–III–V–VI–II), and (II–III–IV–V–VI–II), their corresponding diagonal elements in \mathbf{T}_d are the same and equal 1.08. Node IV participates in the first two cycles so the corresponding diagonal element is different and equals 1.02. Node III participates in the first and third cycle so the corresponding diagonal element is again different and equals 1.065.

IV. PROOF OF INVERTIBILITY AND MATRIX POWER SERIES CONVERGENCE RESULTS IN SYSTEMS WITH AND WITHOUT CIRCULATING FLOWS

In this section, we provide rigorous mathematical proofs of the convergence of the matrix power series (20) in systems with and without circulating flows, and also of the invertibility of the downstream distribution matrix \mathbf{D}_d . This is of fundamental importance for a practical implementation of tracing as it proves the existence and uniqueness of a solution to the tracing problem. For these proofs, the following properties of \mathbf{D}_d are required.

- All column sums are less than or equal to 1.
- For at least one column, labeled K , the column sum is strictly less than 1.
- All elements are nonnegative.

The first and the third assumptions follow directly from the definition of \mathbf{D}_d ; the second assumption is due to a property of \mathbf{D}_d explained in Section II-B. A generator connected to a node means that less than 100% of that node's supply is from previous nodes in the network so that the sum of the nodal shares, i.e., the elements in the corresponding column of \mathbf{D}_d , is less than 1. The second assumption therefore implies that there is at least one source in the network, which is required for a physically realistic power flow.

A. Theorem: All Eigenvalues of \mathbf{D}_d Have Magnitude Less Than 1

Suppose that \mathbf{D}_d has an eigenvector \mathbf{x} with eigenvalue λ . By definition

$$\mathbf{D}_d \mathbf{x} = \lambda \mathbf{x}. \quad (28)$$

Equivalently

$$\lambda x_i = \sum_j D_{ij} x_j. \quad (29)$$

For convenience, the elements of \mathbf{D}_d are written D_{ij} without the subscript d . It follows that

$$\left| \sum_i \lambda x_i \right| = |\lambda| \sum_i |x_i| = \left| \sum_{ij} D_{ij} x_j \right| \leq \sum_{ij} D_{ij} |x_j|. \quad (30)$$

Taking column K out of the sum over j

$$\sum_{ij} D_{ij} |x_j| = |x_K| \sum_i D_{iK} + \sum_{j \neq K} |x_j| \sum_i D_{ij}. \quad (31)$$

By the assumptions above, the first term on the right-hand side is strictly less than $|x_K|$ and the second is no greater than $\sum_{j \neq K} |x_j|$. It follows that

$$|\lambda| \sum_i |x_i| < \sum_i |x_i|. \quad (32)$$

Dividing (32) by $\sum_i |x_i|$, the proof is complete.

B. Corollary: The Matrix Power Series (20) Converges

All eigenvalues having magnitude strictly less than 1 is a sufficient condition for the power series to be convergent [16].

C. Corollary: \mathbf{A}_d Is Invertible

Suppose that \mathbf{A}_d is singular. It follows that an eigenvector \mathbf{x} exists with

$$\mathbf{A}_d \mathbf{x} = (\mathbf{I} - \mathbf{D}_d) \mathbf{x} = 0 \quad (33)$$

and that

$$\mathbf{D}_d \mathbf{x} = \mathbf{x}. \quad (34)$$

As all eigenvalues of \mathbf{D}_d have magnitude less than 1, it follows that \mathbf{A}_d cannot be singular, and that it is therefore invertible.

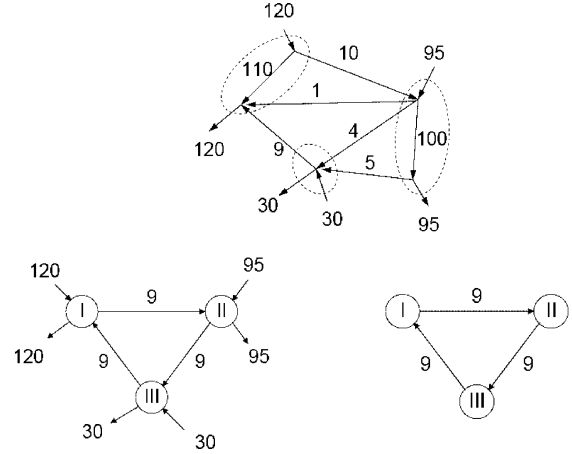


Fig. 8. Possible problems with aggregation of nodes. (a) Original digraph. (b) Aggregated nodes but without netting out of generations/demands. (c) Aggregated nodes with netted out generations/demands.

The above rigorous mathematical proof has an intuitive relationship with the engineering problem. The matrix is invertible and therefore a unique solution to the tracing problem exists when there is at least one source (and therefore also a sink) in the network.

D. Example of Non-Invertible \mathbf{A}_d

Despite its apparently unphysical nature, it is interesting to consider a network with no generation (i.e., a pure circulating flow) when elements in each column of \mathbf{D}_d add up to one. This may in principle arise when nodes are aggregated either to simplify large network diagrams or when individual areas in an interconnected network are aggregated into supernodes.

Consider the acyclic digraph of flows for a full network shown in Fig. 8(a). Fig. 8(b) shows a reduced digraph when two pairs of nodes have been aggregated. This creates a circulating flow, but the tracing solutions still exist as the reduced digraph has sources and sinks. However if the generations and demands are netted out, as shown in Fig. 8(c), the tracing matrices are singular and the problem has no solution. Each of the nodes shown in Fig. 8(c) represents an area with perfectly balanced internal generation and demand; their net injections are zero but circulating flows appear due to geographical imbalance of generation/demand pattern within each area. Note that such a situation cannot arise when the internal generation and demand in each area is explicitly considered, as we have assumed in Figs. 1 and 8(b).

To prove that the solution for the digraph shown in Fig. 8(c) does not exist consider the first five powers of \mathbf{D}_d

$$\begin{aligned} \mathbf{D}_d &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & \mathbf{D}_d^2 &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \mathbf{D}_d^3 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \mathbf{D}_d^4 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & \mathbf{D}_d^5 &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \end{aligned} \quad (35)$$

Note that the sum of all elements in each column of \mathbf{D}_d is equal to 1 as there are no generators in the system. The power series (20) is clearly non-convergent, as the powers of \mathbf{D}_d are periodic, and thus do not tend to zero. Obviously, it is also clear that $\mathbf{A}_d = (\mathbf{I} - \mathbf{D}_d)$ is singular (adding rows shows that they are linearly dependent.)

It is easy to check that the eigenvalues of \mathbf{D}_d are 1 and $(-1 \pm j\sqrt{3})/2$, all of which have magnitude ≥ 1 . This is the consequence of all columns adding up to one. Consequently the conditions required for the proofs above that (20) converges and \mathbf{A}_d is invertible are not satisfied.

E. Conditioning of \mathbf{A}_d

The above considerations make it also possible to analyze how well-conditioned matrix \mathbf{A}_d is. If system generations are small compared to the nodal flows, the nodal shares provided by those generations are small. It follows from the arguments above that in this case \mathbf{A}_d has an eigenvalue close to zero, and hence might be ill-conditioned and hard to invert numerically.

There is an important practical implication of the above finding. As discussed in Section IV-D, internal networks in an interconnected system may be aggregated by netting out the internal injections. Care should be exercised here, as if all countries have roughly balanced generation and demand, net injections would be small while there could be still large cross-border flows due to a geographical imbalance of generation/demand in each area. In the extreme case of net zero nodal injections this would give rise to a pure circulating flow and non-invertible tracing matrices, as discussed in Section IV-D and illustrated in Fig. 8(c).

V. CONCLUSION

This paper has provided an additional insight into the tracing methodology, by representing the inverted original upstream and downstream distribution matrices, $(\mathbf{A}_u)^{-1}$ and $(\mathbf{A}_d)^{-1}$, in the form of matrix power series and by analysing the existence, uniqueness of solution and conditioning of the matrices which must be inverted in the tracing problem.

For the downstream version, $(\mathbf{A}_d)^{-1} = \mathbf{I} + \sum_{n=1}^{\infty} \mathbf{D}_d^n$, where $\mathbf{D}_d = \mathbf{I} - \mathbf{A}_d$. Element $[\mathbf{D}_d^n]_{ij}$ of the n th term in the power series, and the corresponding digraph, show the proportion of the nodal power j which is supplied from node i indirectly through paths of length exactly n , i.e., when power flows through $(n-1)$ intermediate nodes. For acyclic flows, the highest nonzero matrix power has n equal to the height of the original digraph, i.e., the longest path between any two nodes.

The matrix power series representation provides an additional insight into the physical meaning of the inverted tracing distribution matrices. Firstly it proves the link between the graph-based and linear equation-based approaches to electricity tracing, by showing that each element of $(\mathbf{A}_d)^{-1}$ is equal to the sum of contributions to the nodal flows traced back to different nodes in the digraph. Secondly, the matrix power series representation makes it possible to explain an earlier observation that circulating flows, which cause cycles in the digraph of flows, can be detected by appearance of elements greater-than-one on the leading diagonal of the inverted tracing distribution matrices.

The third contribution is probably the most important, as for the first time invertibility of the tracing distribution matrices has been proved rigorously. This is of fundamental importance for a practical implementation of tracing as it proves the existence and uniqueness of a solution to the tracing problem. The proof requires only the presence of generation and demand in the network; systems which do not satisfy this may appear only in some special cases when subsystems in an interconnected network are aggregated to form supernodes. This result implies that the tracing distribution matrices become ill-conditioned when generations are small compared to nodal flows; equivalently, the matrix power series would then converge more slowly.

ACKNOWLEDGMENT

J. W. Bialek would like to thank Prof. F. F. Wu for his support during sabbatical leave at the University of Hong Kong in 2008, when this paper was written.

REFERENCES

- [1] R. Green, "Electricity transmission pricing: An international comparison," *Util. Policy*, vol. 6, no. 3, pp. 177–184, 1997.
- [2] J. Bialek, "Tracing the flow of electricity," *Proc. Inst. Elect. Eng., Gen., Transm., Distrib.*, vol. 143, no. 4, pp. 313–320, Jul. 1996.
- [3] J. W. Bialek and P. Kattuman, "Proportional sharing assumption in tracing methodology," *Proc. Inst. Elect. Eng., Gen., Transm., Distrib.*, vol. 151, no. 4, pp. 526–532, Jul. 2004.
- [4] H. X. Wang, R. Liu, W. D. Li, and C. H. Zhao, "Power flow tracing with consideration of the electrical distance," in *Proc. Asia Pacific Power and Energy Engineering Conf. (APPEEC)*, Wuhan, China, Mar. 28–31, 2009.
- [5] L. Olmos and I. J. Pérez-Arriaga, "Evaluation of three methods proposed for the computation of inter-TSO payments in the Internal Electricity Market of the European Union," *IEEE Trans. Power Syst.*, vol. 22, no. 4, pp. 1507–1522, Nov. 2007.
- [6] A. R. Abhyankar, S. A. Soman, and S. A. Khaparde, "Optimization approach to real power tracing: An application to transmission fixed cost allocation," *IEEE Trans. Power Syst.*, vol. 21, no. 3, pp. 1350–1361, Aug. 2006.
- [7] M. Pantoš, G. Verbič, and F. Gubina, "Modified topological generation and load distribution factors," *IEEE Trans. Power Syst.*, vol. 20, no. 4, pp. 1998–2005, Nov. 2005.
- [8] D. Kirschen, R. Allan, and G. Strbac, "Contributions of individual generators to loads and flows," *IEEE Trans. Power Syst.*, vol. 12, no. 1, pp. 52–60, Feb. 1997.
- [9] F. F. Wu, Y. X. Ni, and P. Wei, "Power transfer allocation for open access using graph theory—Fundamental and applications in systems without loopflow," *IEEE Trans. Power Syst.*, vol. 15, no. 3, pp. 923–929, Aug. 2000.
- [10] J. W. Bialek and S. Ziemianek, "Tracing-based transmission pricing of cross-border trades: Fundamentals and circular flows," in *Proc. IEEE Bologna Power Tech. Conf.*, Bologna, Italy, Jun. 23–26, 2003.
- [11] K. G. Xie, J. Q. Zhou, and W. Y. Li, "Analysis model and algorithm for tracing active power flow based on extended incidence matrix," *Elect. Power Syst. Res.*, vol. 79, no. 2, pp. 399–405, Feb. 2009.
- [12] S. Ziemianek and J. W. Bialek, "Identification of circular flows in transmission networks using the electricity tracing methodology," in *Proc. 16th Power Systems Computation Conf. (PSCC 2008)*, Glasgow, U.K.
- [13] J. W. Bialek, S. Ziemianek, and R. Wallace, "A methodology for allocating transmission losses due to cross-border trades," *IEEE Trans. Power Syst.*, vol. 19, no. 3, pp. 1255–1262, Aug. 2004.
- [14] S. Skiena, *Implementing Discrete Mathematics: Combinatorics and Graph Theory With Mathematica*. Reading, MA: Addison-Wesley, 1990.
- [15] [Online]. Available: <http://www.ee.washington.edu/research/pstca/>.
- [16] G. H. Golub and C. F. Van Loan, *Matrix Computations*. Baltimore, MD: John Hopkins Univ. Press, 1983, ch. 11.
- [17] P. Wei, Y. X. Ni, and F. F. Wu, "Load flow tracing in power systems with circulating power," *Int. J. Elect. Power Energy Syst.*, vol. 24, no. 10, pp. 807–813, Dec. 2002.



Chira Achayuthakan received the B.Eng. degree in 1995 from Chulalongkorn University, Bangkok, Thailand, and the M.Eng. degree in 1997 from the Asian Institute of Technology (AIT), Pathumthani, Thailand, where he is currently pursuing the Ph.D. degree in energy field of study.

Before he joined the Electricity Generating Authority of Thailand (EGAT), he worked as O&M Shift Charge for Independent Power (Thailand) Company or IPT, one of the IPP pioneers in Thailand. IPT holds a 700-MW combined cycle power

plant. His research interests include energy economics, and deregulated electric supply industry.



Chris J. Dent (M'08) received the B.A. degree in mathematics from Cambridge University, Cambridge, U.K., in 1997, the Ph.D. degree in theoretical physics from Loughborough University, Loughborough, U.K., in 2001, and the M.Sc. degree in operational research from the University of Edinburgh, Edinburgh, U.K., in 2006.

He is a Senior Research Associate in the School of Engineering and Computing Sciences, Durham University, Durham, U.K. From 2007–2009, he was with the University of Edinburgh. His research inter-

ests lie power system optimization, risk modelling, economics, and renewables integration.

Dr. Dent is a Chartered Physicist, an Associate of the OR Society, and a member of the Institution of Engineering and Technology, the Operational Research Society, and Cigre.



Janusz W. Bialek (SM'08) received the M.Eng. and Ph.D. degrees from Warsaw University of Technology, Warsaw, Poland, in 1977 and 1981, respectively.

He is a Professor of electrical power and control at Durham University, Durham, U.K. In 1981–1989, he was with Warsaw University of Technology, in 1989–2002 with Durham University, and in 2003–2009 with the University of Edinburgh. He coauthored two books and over 100 technical papers. His research interests include power system

economics, power system dynamics, and sustainable energy systems.



Weerakorn Ongsakul (S'89–M'95) received the B.Eng. degree from Chulalongkorn University, Bangkok, Thailand, in 1988 and the M.S. and Ph.D. degrees in electrical engineering from Texas A&M University, College Station, in 1991 and 1994, respectively.

He is currently an Associate Professor of energy field of study and Dean of the School of Environment, Resources, and Development, Asian Institute of Technology, Pathumthani, Thailand. His research interests are in computer applications to power sys-

tems, parallel processing applications, AI applications to power system, and power system deregulation.